Math 112: Introductory Real Analysis

§ Lecture 4 (Feb 5, 2025)

We'll talk about
- Basic (point set) topology in the next few lectures.

Today: Cardinality of sets

Def Let A and B be two sets and $f: A \to B$ a function.

We say that f is injective (or 1-1) if, for each $y \in B$, $f^{-1}(y)$ consists of at most one element,

and that f is surjective (or onto) if, for each $y \in B$, $f^{-1}(y)$ is non-empty.

We say f is bijective (or that f is a 1-1 correspondence) if it is both injective and surjective.

We say that A and B have the same cardinality if there is a 1-1 correspondence between the two.

Let's write $A \sim B$ if A and B have the same cardinality.

Then, (reflexivity) $A \sim A$ for any set A

(symmetry) If $A \sim B$, then $B \sim A$

(transitivity) If $A \sim B$ and $B \sim C$, then $A \sim C$.

In other words, $\sim$ is an equivalence relation.

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The equivalence classes of sets of same cardinality are called cardinal numbers.

Cardinal numbers are (totally) ordered: (assuming Axiom of Choice)

$A \leq B$ if there is an injective map $A \to B$. This is an ordering by Schröder-Bernstein theorem.

$$0 < 1 < 2 < 3 < \cdots < \aleph_0 < \aleph_1 < \cdots$$

Def For each $n \in \mathbb{N}$, let $J_n := \{m \in \mathbb{Z}_{>0} \mid m \leq n\} = \{1, 2, \cdots, n\}$, and $J := \mathbb{Z}_{>0}$.

For any set A, we say

(a) A is finite if $A \sim J_n$ for some $n \in \mathbb{N}$

(b) A is infinite if A is not finite

(c) A is countably infinite if $A \sim J$.

elements of any countably infinite set can be arranged in a sequence.

(d) A is (at most) countable if A is either finite or countably infinite.

(e) A is uncountable if A is not (at most) countable.

E.g. $\mathbb{Z}$ is countably infinite. So are $\mathbb{Z} \setminus \{0\}$ and $\mathbb{Q}$.

$\mathbb{R}$ is uncountable.

Thm Every infinite subset of a countably infinite set is countably infinite.

proof) Suppose $E \subseteq A$, and E is infinite and A is countable.

Arrange the elements of A in a sequence $\{x_n\}_{n=1}^{\infty}$ of distinct elements.

Construct a sequence $\{x_{n_k}\}_{k=1}^{\infty}$ by choosing

n_1 = smallest positive integer such that $x_{n_1} \in E$,

and having chosen $n_1, \cdots, n_{k-1}$,

let n_k = smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Then, $\begin{aligned} J &\longrightarrow E \\ k &\longmapsto x_{n_k} \end{aligned}$ is a 1-1 correspondence ■

Thm Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of countably infinite sets.

Then $S := \bigcup_{n=1}^{\infty} E_n$ is countably infinite.

proof) Arrange each E_n in a sequence $\{x_{n,k}\}_{k=1}^{\infty}$.

Construct a new sequence by

$$\begin{array}{ccccc} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & \cdots \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & \cdots \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & \cdots \\ \cdots & & & & \end{array}$$

i.e. $x_{1,1}$; $x_{2,1}, x_{1,2}$; $x_{3,1}, x_{2,2}, x_{1,3}$; $\cdots$

This shows S is at most countable.

Since $E_1 \subset S$, S is also countably infinite ■

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Thm Let A be a countable set. Then A^n, the set of all n-tuples $(a_1, \cdots, a_n)$ where $a_k \in A$ $(k=1,\cdots,n)$ is also countable.

proof) That $A^1 = A$ is countable is evident.

Suppose A^{n-1} is countable.

Then $A^n = A^{n-1} \times A$ is a countable union of countable sets, and hence A^n is countable.

The theorem follows by induction.

Cor $\mathbb{Q}$ is countable.

proof) Every rational number is of the form $\frac{b}{a}$, $a, b \in \mathbb{Z}$, and the set of pairs $(a,b) \in \mathbb{Z}^2$ is countable.

Therefore, the set of fractions $\frac{b}{a}$ is also countable.

Thm (Cantor's diagonal argument)

For every set S, the power set $\mathcal{P}(S) = 2^S := \{E \subset S\}$ has cardinality greater than S.

proof) Suppose there were a surjective function $f: S \to \mathcal{P}(S)$.

Consider the set $T := \{s \in S \mid s \notin f(s)\}$.

Then T cannot be in the image of f, as if it were, $T = f(s)$ for some $s \in S$, but $\begin{cases} s \in T \Rightarrow s \notin f(s) = T \\ s \notin T \Rightarrow s \in f(s) = T \end{cases}$

so contradiction ■

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<u>Cor</u> $\mathbb{R}$ is uncountable.

proof) Every real number has a binary expansion (i.e. can be represented by a sequence of 0's and 1's), and the set of all sequences in 0's and 1's (i.e. 2^{J_0}) is uncountable.

Therefore, $\mathbb{R}$ is uncountable ■

More explicitly, for any function $J \to \mathbb{R}$

binary expansions $s_{i,j} \in \{0,1\}$

$$\begin{aligned} 1 &\longmapsto *\cdots*.\,s_{11}\,s_{12}\,s_{13}\cdots \\ 2 &\longmapsto *\cdots*.\,s_{21}\,s_{22}\,s_{23}\cdots \\ &\ \vdots \end{aligned}$$
,

the real number $0.s_1' s_2' s_3' \cdots$ where $s_i' := 1 - s_{ii}$

is <u>not</u> in the image.